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On Impatience, Temptation & Ramsey's Conjecture [*]

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Abstract

This article is aimed at exploring the implications of the introduction of self-control and temptation motives in inter temporal preferences within an elementary competitive equilibrium with production. Letting heterogeneous agents differ from both their discounting parameters and their temptation motives, this article is interested in the long-run distribution of consumptions and wealths. Results are at odds from the ones obtained in a standard Ramsey benchmark setup in that long-run distributions are commonly non degenerated ones.

Keywords: Impatience, Temptation, Self-Control, Ramsey's Conjecture.

JEL Classification: E32, O41.

Contents

I. Introduction

Within a breakthrough contribution, Gul & Pesendorfer [4] has come to introduce a new model of inter temporal choices based upon preferences with an interaction between a temptation motive directed towards immediate consumption and a long-run concern that springs from the discounting the future. More explicitly, they built a new axiomatisation of preferences mainly based upon three distinct assumptions. Firstly, preferences will not only depend upon the actual consumption sequences, but also upon the set of all admissible trajectories. Secondly, a set betweeness axiom pictures a preference for commitment and self-control. Finally, resting upon an axiom of temptation for an immediate consumption plus a list of more standard axioms, a new temptation-based representation theorem for preferences is established. More explicitly, for x_t that denotes the quantity of wealth held by an individual agent and c_t that denotes his consumption, letting $\mathrm{B}_t(x_t)$ feature the set of reachable levels of consumption and wealth along: $(c_t, x_{t+1}) \in \mathrm{B}_t(x_t)$, preferences are then to be represented through a function J_t that satisfies:

$$\mathrm{J}_t(x_t) := \max_{(c_t, x_{t+1}) \in \mathrm{B}_t(x_t)} \Big\{ u(c_t) + \gamma v(c_t) + \delta \mathrm{J}_{t+1}(x_{t+1}) \Big\} - \max_{(c_t, x_{t+1}) \in \mathrm{B}_t(x_t)} \Big\{ \gamma v(c_t) \Big\},$$

for $\delta \in]0,1[$ the discounting parameter and $\gamma \geqslant 0$ the temptation parameter. Through his very choice, an agent will complete an arbitrage between his long-run welfare and the short-run temptation of a maximal amount of immediate consumption. It is worth emphasizing that, with respect to the standard model of inter temporal choice, the Gul and Pesendorfer model introduces a function $v(\cdot)$ and a parameter γ that will jointly illustrate the temptation motives of the agent.

This contribution considers a basic model with heterogenous individuals where preferences are defined through similar functions $u(\cdot)$ and $v(\cdot)$, but differ from the parameters δ and γ. It will then specialize the analysis to a comprehensive account of the long-run distributions of consumption and wealth. In this regard, it is to be recalled that the decentralized formulation of the standard growth model with heterogenous agents has been the core of numerous contributions dating from the seminal article of Ramsey [9]. The so-called associated Ramsey's conjecture, that was formally established by Becker [1], introduces the following result. Within a model with infinitely-lived agents that differ through their discount rates, only the most patient of the agents benefits from a positive long-run amount of consumption, the other agents being constrained to have nil levels of consumption. Capital therein corresponds to a modified golden rule defined from the discount factor of the most patient amongst the agents. Further, solely the most patient of the agents is to hold a long-run positive amount of consumption. The Ramsey [9] model was then augmented by Becker [1], Le Van & Vailakis [6] and Mitra & Sorger [7] through the introduction of imperfect financial markets that single out the impossibility of indebtedness for the individuals. Within such an augmented framework, the most patient amongst the agents holds the whole stock of capital of the economy and benefits from a level of consumption that exceeds his wage payments. In contradistinction with this, all of the other agents have a level of consumption that boils down to their wage payments and do not hold any capital units. Finally, the level of the capital stock results from the modified golden rule that corresponds to the discount rate of the most patient amongst the agents. For both of these cases, the model eventually leads to degenerated distributions of consumption and wealth.

Under the assumption of a Gul and Pesendorfer's class of preferences for the agents, the long-run features of the heterogenous agents reveal as being strongly modified. First, under a configuration with perfect financial markets and even though they differ from their discount rates, all of the agents may display long-run positive levels of consumption. Interestingly, under a configuration where their temptation motives will be lower than the one of the most patient agent, their levels of long-run consumption could even overcome the one of the benchmark individual. Considering then an economy with imperfect financial markets, this results in a direct modification of the preferences of the agents. The maximum attainable level of consumption by individuals who succumb to temptation is indeed accordingly modified. In the long run, constrained agents will consume their wage whilst the unconstrained agents will benefit from a greater level of consumption. The very fact of being constrained depends upon both the discounting and temptation parameters. As a simple illustration, would the most patient amongst the agents be equally associated with an exceedingly big level of temptation motives, he could well end up as being constrained. To sum up, this simple economy with a Gul and Pesendorfer's class of preferences for the heterogeneous individuals will result in much more satisfactory long-run properties, the distributions of consumption and wealth being far less degenerated than in the standard case.

The whole model is specified with a variable degree of imperfection of the financial markets that can be represented through a parameter λ, for $\lambda = 1$ that corresponds to perfect financial markets whilst $\lambda = 0$ corresponds to the prohibition of any debt. Between these two extreme values, one is then in position to represent a whole range of intermediary scenarios with a limited indebtedness for all of the agents. One can prove that increasing the imperfection of financial markets results into an increase of the long-run stock of capital. Similarly, the long-run capital stock commonly increases with the level of patience of the agents but decreases with their temptation motives. A final part is devoted to the characterisation of the social optimum of the economy. For the optimum, it is always preferable to cancel out the temptation motive. This interestingly results in the fact that, considering a decentralization of such an optimum, it will always be necessary to constrain the level of consumption of the individuals in order not to exceed the optimum level. It is further proved that the imposition of such a constraint, together with a redistribution of the initial wealths, suffices for decentralizing this optimum. Within a world where individuals are subject to temptation motives, the only way to decentralize the optimum is then to forbid the freedom of the consumption choice. The optimality is recovered from one of the first very principles of communism: to anyone according to his needs.

Section II introduces the benchmark framework and completes the definition of the equilibrium. Section III characterizes the long-run equilibria with and without financial constraints. Section IV analyses the comparative statics properties and studies some welfare topics.

II. General Equilibrium of an Economy with Heterogeneous Agents

II.1 A Setup and a Class of Preferences

Time is discrete. Consider a competitive economy with n infinitely-lived individuals indexed by $i = 1, 2, \ldots, n$. Let then $\{c_t^i\}_{t \geqslant 0}$ denote the sequence of the consumptions of individual $i = 1, 2, \ldots, n$ whilst $\{w_t\}_{t \geqslant 0}$ and $\{R_t\}_{t \geqslant 0}$ respectively stand for the wage and gross interest rates sequences he faces to, for $R_t = 1 + r_t$, where r_t denotes the rate of interest. Let then x_t^i denote the wealth held by the agent i at date $t \geqslant 0$, $x_t^i = R_t a_t^i + w_t$, where a_t^i features the capital held at date $t \geqslant 0$. His budget constraint eventually writes down at date $t \geqslant 0$ along:

$$x_{t+1}^i = R_{t+1}(x_t^i - c_t^i) + w_{t+1}$$

An individual agent is further subject to an inter temporal budget constraint that is defined over his lifespan. Assuming that $\sum_{t=0}^{+\infty} w_t / \prod_{\tau=0}^{t} R_\tau \leqslant +\infty$, his discounted future human wealth at date $t \geqslant 0$ is $h_{t+1}^i = \sum_{j=1}^{+\infty} w_{t+j} / \prod_{\tau=1}^{j} R_{t+\tau}$ and happens to be the same for any agent $i = 1, 2, \ldots, n$. The lifespan budget constraint of an agent $i = 1, 2, \ldots, n$ will eventually formulate along

$$c_0^i + \sum_{t=1}^{+\infty} \Big(c_t^i \,/\, \prod_{\tau=1}^{t} R_\tau \Big) \leqslant x_0^i + h_1,$$

for h_1 that denotes his discounted future human wealth by the initial date. The agent will further face at any period a financial constraint:

$$c_t^i \leqslant x_t^i + \lambda h_{t+1}, \quad \lambda \in [0, 1].$$

Within the above expression, λ denotes the fraction of human wealth that the agent may consume. The occurrence of $\lambda = 1$ will then correspond to the omission of any liquidity constraint and perfect financial markets: an agent will be in position to consume the whole amount of his lifespan wealth. In opposition to this, the occurrence of $\lambda = 0$ will correspond to a configuration for which the agent willn't be allowed to borrow and face imperfect financial markets.

Following Gul & Pesendorfer, the agent's problem formulates according to:

$$J_t^i(x_t^i) = \max_{\substack{c_t^i \in [0, x_t^i + \lambda h_{t+1}], \\ x_{t+1}^i = R_{t+1}(x_t^i - c_t^i) + w_{t+1}}} \Big\{ u(c_t^i) + \gamma_i [v(c_t^i) - v(x_t^i + \lambda h_{t+1})] + \delta_i J_{t+1}^i(x_{t+1}^i) \Big\},$$

Assumption P1. For $\gamma_i \geqslant 0$, $u(\cdot)$ and $v(\cdot)$ are defined on $\mathbb{R}_+$, of class $\mathscr{C}^2$, with[1] $Du(c^i) > 0$, $D^2 u(c^i) < 0$, $Dv(c^i) > 0$, $D^2 v(c^i) \geqslant 0$, $D^2 u(c^i) + \gamma_i D^2 v(c^i) < 0$ for $c^i > 0$ and where $\lim_{c^i \to 0} Du(c^i) = +\infty$, $\lim_{c^i \to \infty} Du(c^i) = 0$.

Within the above formulation, the non-positive component $\gamma_i[v(c_t^i) - v(x_t^i + \lambda h_{t+1})]$ pictures the welfare loss when one withstands the temptation of consuming the whole amount. It is then worthwhile noticing that this component cancels out when the individual oppositely succumbs to the temptation. The integration of the parameter $\gamma_i \geqslant 0$ in the preferences of the individual allows for letting the weight of temptation vary whilst $\delta_i \in]0, 1[$ describes the psychological discount factor. Otherwise stated, this class of preferences happens to exhibit a dependency with respect to the maximal level of consumption. The introduction of a liquidity constraint results in a direct modification of the preferences. Finally, it is worth emphasizing that these preferences reformulate along:

$$\sum_{t=0}^{+\infty} (\delta_i)^t \{ u(c_t^i) + \gamma_i [v(c_t^i) - v(x_t^i + \lambda h_{t+1})] \}.$$

[1] $Du(c) := \partial u(c) / \partial c$.

The consumer problem lists along:

$$(1)\qquad \max_{\{c_t^i, x_{t+1}^i\}_{t\geqslant 0}} \sum_{t=0}^{+\infty} (\delta_i)^t \{u(c_t^i) + \gamma_i [v(c_t^i) - v(x_t^i + \lambda h_{t+1})]\}$$
$$\text{s.t. } x_{t+1}^i = R_{t+1}(x_t^i - c_t^i) + w_{t+1},$$
$$0 \leqslant c_t^i \leqslant x_t^i + \lambda h_{t+1},$$
$$x_0^i \text{ given}$$

Under Assumption P.1, the problem is concave as it builds from an objective function that is concave as a function of c_t and concave with respect to $x_t^i + \lambda h_{t+1}$. The holding of $c_t^i \geqslant 0$ always prevails and the following conditions are thus necessary and sufficient for optimality:

(2a) $Du(c_t^i) + \gamma_i Dv(c_t^i) = \delta_i \mu_{t+1}^i R_{t+1} + \nu_t^i,$

(2b) $\mu_t^i = \delta_i \mu_{t+1}^i R_{t+1} - \gamma_i Dv(x_t^i + \lambda h_{t+1}) + \nu_t^i,$

(2c) $\lim_{t\to+\infty} (\delta_i)^t \mu_t^i x_t^i = 0,$

for μ_t^i and ν_t^i that denote the shadow prices respectively associated with the constraints $x_{t+1}^i = R_{t+1}(x_t^i - c_t^i) + w_{t+1}$ and $c_t^i \leqslant x_t^i + \lambda h_{t+1}$. Taking differences between the two first equations delivers $\mu_t^i = Du(c_t^i) + \gamma_i[Dv(c_t^i) - Dv(x_t^i + \lambda h_{t+1})]$. At some date $t \geqslant 0$, an agent can be constrained or unconstrained. An unconstrained individual has a consumption level c_t^i that satisfies $c_t^i < x_t^i + \lambda h_{t+1}$ whilst $\nu_t^i = 0$ prevails. In opposition to this, a constrained agent by date $t \geqslant 0$ is singled out by a consumption level c_t^i such that $c_t^i = x_t^i + \lambda h_{t+1}$ whilst $\nu_t^i > 0$.

It is further assumed that:

Assumption P2. $1 > \delta_1 > \delta_2 > \cdots > \delta_n$.

This economy encompasses a competitive representative firm with a production function $F(K_t, L_t)$, for K_t the capital stock at date $t \geqslant 0$ that depreciates at the rate $\eta \in]0,1[$ and L_t the quantity of labour at that same date. Any of the individuals inelastically offers $1/n$ units of labour, so that the global amount that is offered summarises to 1. The average wage at date $t \geqslant 0$ is hence available as $w_t = (1/n) D_L F(K_t, 1)$ for any of the individuals.

Assumption T1. $F(K, L)$ is a strictly increasing function that is concave, of class $\mathscr{C}^2$, homogeneous of degree one and such that $D_K F(0,1) = +\infty$, $D_K F(+\infty, 1) < \eta$.

Under Assumption T.1, there exists a critical level of the capital stock $\hat{K}$ such that $D_K F(\hat{K}, 1) = \eta$, for $\hat{K}$ that corresponds to the golden rule of accumulation.

II.2 Equilibrium

Definition II.1. An inter temporal equilibrium of the economy populated by the n agents, $i = 1, \ldots, n$, is defined by a sequence $\{c_t^i, x_t^i, h_t, K_t, w_t, R_t, \mu_t^i, \nu_t^i\}_{t\geqslant 0}$ such that, for any $t \geqslant 0$ and for any $i = 1, 2, \ldots, n$:

(3a) $x_{t+1}^i = R_{t+1}(x_t^i - c_t^i) + w_{t+1},$

(3b) $Du(c_t^i) + \gamma_i Dv(c_t^i) = \delta_i \mu_{t+1}^i R_{t+1} + \nu_t^i,$

(3c) $\mu_t^i = \delta_i \mu_{t+1}^i R_{t+1} - \gamma_i Dv(x_t^i + \lambda h_{t+1}) + \nu_t^i$,

(3d) $\lim_{t\to+\infty} (\delta_i)^t \mu_t^i x_t^i = 0$,

(3e) for $\nu_t^i > 0, c_t^i = x_t^i + \lambda h_{t+1}$,

(3f) for $\nu_t^i = 0, c_t^i < x_t^i + \lambda h_{t+1}$,

(3g) $$h_{t+1} = \sum_{j=1}^{+\infty} \frac{w_{t+j}}{\prod_{\tau=1}^{j} R_{t+\tau}},$$

(3h) $w_t = (1/n) D_L F(K_t, 1)$,

(3i) $R_t = D_K F(K_t, 1) + (1 - \eta)$,

(3j) $$\sum_{i=1}^{n} \frac{x_t^i - w_t}{R_t} = K_t.$$

III. Long-Run Equilibrium of an Economy with Heterogeneous Agents

III.1 Characterisation

Proposition III.1.— *A stationary competitive equilibrium of the economy with heterogenous agents is characterized by consumption levels* $(c^1, c^2, \dots, c^n)$ *together with an aggregate capital stock given by* K, *factor prices given by* w *and* R, *that satisfy*

(4a) $$\sum_{i=1}^{n} c^i = F(K, 1) - \eta K,$$

(4b) $R = D_K F(K, 1) + (1 - \eta), \quad w = (1/n) D_L F(K, 1)$,

(i) *if* $\gamma_i\{Dv((1-\lambda)w)/Du((1-\lambda)w)\} \geqslant \delta_i R - 1$, *then* $c^i = (1-\lambda)w$;

(ii) *if* $\gamma_i\{Dv((1-\lambda)w)/Du((1-\lambda)w)\} < \delta_i R - 1$, *then* c^i *satisfies* $c^i > (1-\lambda)w$ *and is defined from*

(5) $$\frac{\delta_i R}{\delta_i R - 1} \gamma_i Dv\left(\frac{Rc^i - (1-\lambda)w}{R - 1}\right) = Du(c^i) + \gamma_i Dv(c^i).$$

The first population of individuals, described through Proposition III.1(i), are constrained and their consumption is available as $c^i = (1-\lambda)w$. The second population of individuals, described through Proposition III.1(ii), are unconstrained and their consumption satisfies (5). As assessed through the following statement, this results into a consumption function $c^i = C(\delta_i, \gamma_i, R, w, \lambda)$:

Lemma III.1.—[Unconstrained Consumption] *Consider a stationary solution for the unconstrained individual such that:*

(6) $$\frac{\gamma_i Dv[(1-\lambda)w]}{Du[(1-\lambda)w]} < \delta_i R - 1.$$

(i) *The equation* (5) *determines a unique stationary consumption solution* $c^i = C(\delta_i, \gamma_i, R, w, \lambda) > (1-\lambda)w$.

(ii) *The unconstrained stationary consumption* $c^i = \mathrm{C}(\delta_i, \gamma_i, \mathrm{R}, w, \lambda)$ *is non-decreasing in* w, *increasing in* R *and* δ_i, *decreasing in* γ_i *and non-increasing in* λ. *Moreover,* $\lim_{\mathrm{R}\to+\infty}\mathrm{C}(\delta_i, \gamma_i, \mathrm{R}, w, \lambda) = +\infty$.

(iii) *For* $\lambda = 1$ *and with perfect financial markets,* $\mathrm{C}(\delta_i, \gamma_i, \mathrm{R}, w, 1)$ *is independent of* w.

It is noticed that, for perfect financial markets and $\lambda = 1$, the inequation (6) becomes $\delta_i \mathrm{R} > 1$. The unconstrained individual consumption function C can be perceived as the consumption resulting from a partial equilibrium argument with w and R exogenously given.

III.2 An Economy with Perfect Financial Markets

This section will establish the existence and the uniqueness of the steady state when financial markets are perfect.

Definition III.1. A stationary competitive equilibrium of the economy with heterogenous agents and perfect financial markets is characterized by consumption levels $(c^1, c^2, \ldots, c^n)$ with an aggregate capital stock given by K, factor prices given by w and R that satisfy

$$\sum_{i=1}^{n} c^i = \mathrm{F}(\mathrm{K}, 1) - \eta \mathrm{K},$$
$$\mathrm{R} = \mathrm{D_K F}(\mathrm{K}, 1) + (1-\eta), \quad w = (1/n)\mathrm{D_L F}(\mathrm{K}, 1),$$

(i) if $\delta_i \mathrm{R} > 1$, then $c^i > 0$ and is defined from $c^i = \mathrm{C}(\delta_i, \gamma_i, \mathrm{R}, w, \lambda)$;

(ii) if $\delta_i \mathrm{R} \leq 1$, then $c^i = 0$.

Letting henceforward $\mathrm{R}(\mathrm{K}) := \mathrm{D_K F}(\mathrm{K}, 1) + (1-\eta)$ and $w(\mathrm{K}) := \mathrm{D_L F}(\mathrm{K}, 1)$, the frontier between constrained and unconstrained behaviours defines, for any agent $i = 1, 2, \ldots, n$, a value $\tilde{\mathrm{K}}_i$ such that

$$\mathrm{R}(\tilde{\mathrm{K}}_i) = 1/\delta_i.$$

Agent i will then have a positive consumption for $\mathrm{K} < \tilde{\mathrm{K}}_i$ and a nil amount of consumption for $\mathrm{K} \geqslant \tilde{\mathrm{K}}_i$. At that stage, it is worth emphasizing that the ranking of the elements of $\{\tilde{\mathrm{K}}_1, \ldots, \tilde{\mathrm{K}}_n\}$ is a direct byproduct of the one associated to the δ_i, $i = 1, 2, \ldots, n$ under Assumption P.2: $\tilde{\mathrm{K}}_1 > \tilde{\mathrm{K}}_2 > \ldots > \tilde{\mathrm{K}}_n$. From Lemma III.1 and the expression of factor prices $\mathrm{R}(\mathrm{K})$ and $w(\mathrm{K})$, the stationary amount of consumption c^i of agent i can be expressed as a function of K along:

$$\begin{aligned} c^i &= \mathrm{C}(\delta_i, \gamma_i, \mathrm{R}(\mathrm{K}), w(\mathrm{K}), 1) \\ &= \mathscr{C}(\delta_i, \gamma_i, \mathrm{K}, 1). \end{aligned}$$

Moreover, C being independent of $w(\mathrm{K})$ under perfect financial markets and for $\lambda = 1$, this results in $\mathscr{C}$ that is a decreasing function of K for any $\mathrm{K} \in]0, \tilde{\mathrm{K}}_i[$.

Proposition III.2.— *Under the previous set of assumptions:*

(i) *there does exist a stationary competitive equilibrium for the economy characterized by a value* $\mathrm{K}^* \in]0, \tilde{\mathrm{K}}_1[$ *and it is unique;*

(ii) *letting* $j \in \{1, \dots, n\}$ *be such that* $\tilde{\mathrm{K}}_{j+1} \leqslant \mathrm{K}^* < \tilde{\mathrm{K}}_j$, *agents* $1, \dots, j$ *with benefit from positive stationary consumption amounts* $c^i = \mathscr{C}(\delta_i, \gamma_i, \mathrm{K}, 1)$ *whereas the consumptions of agents* $j+1, \dots, n$ *will sum up to zero.*

The following examples will show that the long-run distribution of consumptions and wealths may display features that stringently differ from the traditional Ramsey setup.

Example III.1. Within an economy with perfect financial markets $(\lambda = 1)$ and for agents that merely differ from their discount rate, the stationary distribution of wealth for the individuals may be non degenerated. A simple example should illustrate this scope for such a non-degenerated distribution. Assume that n individuals are characterized by different discounting parameters δ_i, with $\bar{\delta} := \sum_{i=1}^n \delta_i / n$ that describes the average discount factor. In order to simplify matters, it is further assumed that all of the individuals are characterized by the same temptation parameter γ. It is finally assumed that $u(c) = \ln(c)$ and $v(c) = c$. When the level of consumption of an individual is positive at a steady state, it is available as $c^i = (\mathrm{R}\delta_i - 1)/\gamma$. The aim is then to build an equilibrium for which all of the individuals, even though they differ from their discount rates, benefit from a positive amount of consumption. Such an equilibrium satisfies

$$\sum_{i=1}^{n} c^i = \mathrm{F}(\mathrm{K}, 1) - \eta \mathrm{K},$$

for $c^i = (\mathrm{R}\delta_i - 1)/\gamma$ and $\mathrm{R} = \mathrm{D_K F}(\mathrm{K}, 1) + (1 - \eta)$, or

$$\sum_{i=1}^{n} [(\mathrm{D_K F}(\mathrm{K}, 1) + (1 - \eta))\delta_i - 1]/\gamma = \mathrm{F}(\mathrm{K}, 1) - \eta \mathrm{K},$$

that restates along:

$$\Big[(\mathrm{D_K F}(\mathrm{K}, 1) + (1 - \eta))\bar{\delta} - 1\Big] n/\gamma = \mathrm{F}(\mathrm{K}, 1) - \eta \mathrm{K}.$$

This equation has a unique solution $\mathrm{K}^* < \hat{\mathrm{K}}$ such that $(\mathrm{D_K F}(\mathrm{K}^*, 1) + (1-\eta))\bar{\delta} > 1$. Consider then $\delta_n := \min_i(\delta_i)$. Under the holding of $\delta_n > \delta_\ell$, for δ_ℓ that is defined from $(\mathrm{D_K F}(\mathrm{K}^*, 1) + (1 - \eta))\delta_\ell = 1$, and even though they are characterized by distinct values of δ_i, all of the agents benefit from a positive level of consumption.

Example III.2. Consider an economy with perfect financial markets that is populated by two individuals such that $\delta_1 > \delta_2$ and $\gamma_1 > \gamma_2$. Interestingly, one may exhibit stationary states parameter configurations for which $c^1 < c^2$, i.e., the most patient of the agents will assume a lower amount of stationary consumption. Keeping, along Example III.1, on retaining preferences described by $u(c) = \ln(c)$ and $v(c) = c$, the steady state value of the capital stock does correspond to the unique solution of the following equation:

$$\Big[\mathrm{D_K F}(1, \mathrm{K}) + 1 - \eta\Big] \sum_{i=1}^{2} \frac{\delta_i}{\gamma_i} - \sum_{i=1}^{2} \frac{1}{\gamma_i} = \mathrm{F}(\mathrm{K}, 1) - \eta \mathrm{K}.$$

Letting then $b = \sum_{i=1}^{2}(1/\gamma_i)$ and $a = \sum_{i=1}^{2}(\delta_i/\gamma_i)$ and let the values of γ_i, $i = 1, 2$ be given. Consider variations of $\delta_i, i = 1, 2$ that will leave unmodified the value of the coefficient a and in turn result in unmodified values for the capital stock.

More precisely consider the variations of δ_1 and δ_2 around a value of $\delta = a/b$ and let $\delta_1 = \delta + x$ for $x > 0$. One then infers δ_2 from the holding of $\delta_2/\gamma_2 = a - \delta_1/\gamma_1$, whence

$$\begin{aligned}\delta_2/\gamma_2 &= \delta b - (\delta + x)/\gamma_1\\ &= \delta/\gamma_2 - x/\gamma_1.\end{aligned}$$

The consumptions of the two individuals become available as:

$$\begin{aligned}c^1 &= \Big(R\delta_1 - 1\Big) \,/\, \gamma_1\\ &= \Big[R(\delta + x) - 1\Big] \,/\, \gamma_1,\\ c^2 &= \Big(R\delta_2 - 1\Big) \,/\, \gamma_2\\ &= R\big(\delta/\gamma_2 - x/\gamma_1\big) - 1/\gamma_2.\end{aligned}$$

Upon a variation of x, R will keep on assuming the same value, for K is unmodified. From the expressions above and for $x = 0$, it is firstly obtained that $c^1 < c^2$, that results from $\delta_1 = \delta_2 = \delta$ and $\gamma_1 > \gamma_2$. Similarly, for $x > 0$ and as long as x assumes sufficiently small orders, one also recovers $c^1 < c^2$ in spite of the holding of $\delta_1 > \delta_2$; the most patient agent will thus be associated with a lower consumption amount.

III.3 An Economy with Imperfect Financial Markets

This part establishes the existence of an equilibrium in the economy with imperfect financial markets.This proceeds from first characterising the frontier between constrained and unconstrained behaviours. Consider, for $\lambda \neq 1$, the equilibrium value for K that corresponds, for an agent i, to the frontier between constrained and unconstrained behaviors:

$$(6)\qquad \gamma_i Dv\big((1-\lambda)w(K)\big) = \Big\{\delta_i[R(K)] - 1\Big\} Du\big((1-\lambda)w(K)\big)$$

Lemma III.2.— [Frontier between the regimes] *Letting* $\tilde{K}_i$ *be uniquely defined from* $R(\tilde{K}_i, 1) = 1/\delta_i$:

(i) *the equation* (6) *of the frontier between constrained and unconstrained regimes determines a unique solution* $K = \check{K}_i(\delta_i, \gamma_i, \lambda)$, for $\check{K}_i(\delta_i, \gamma_i, \lambda) \in]0, \tilde{K}_i[$;

(ii) $\check{K}_i(\delta_i, \gamma_i, \lambda)$ *increases with* δ_i, *decreases with* γ_i *and increases with* λ, *for* $\lim_{\lambda \to 1} \check{K}_i(\delta_i, \gamma_i, \lambda) = \tilde{K}_i$.

Lemma III.2 allows for introducing n values $\check{K}_i$, the agents being henceforward ranked according to the following order:

$$\check{K}_1 > \check{K}_2 > \cdots > \check{K}_n.$$

In contradistinction with the previous section and the prevailing of $\lambda = 1$, the ranking of the individuals as described by $\check{K}_i$, $i = 1, 2, \ldots, n$ depends upon both δ_i and γ_i. From Lemma III.2, the following occurrences are available for the agent i:

— for $K \geqslant \check{K}_i$, $c^i = (1-\lambda)w$ and the agent is constrained;

— for $K < \check{K}_i$, $c^i > (1-\lambda)w$ and the agent is unconstrained.

From a general equilibrium perspective and for a constrained individual, consumption listing as

$$c^i = (1-\lambda)w = (1-\lambda)D_L F(K,1)/n,$$

it emerges as an increasing function of K. It is noteworthy that the configuration $\lambda = 1$ drastically simplifies matters in this regard in imposing $c^i = 0$ for any constrained individual.

Matters are more subtle for an unconstrained individual i whose consumption as a function of K is given by $c^i = \mathscr{C}(\delta_i, \gamma_i, K, \lambda) = C(\delta_i, \gamma_i, R(K), w(K), \lambda)$. The expression $C(\delta_i, \gamma_i, R(K), w(K), \lambda)$ being, from Lemma III.1, increasing as a function of R and non-increasing as a function of w, the resulting effect of a variation of K on the agent's consumption is not determined. Nevertheless, and as this is assessed through the following statement, the function $\mathscr{C}(\delta_i, \gamma_i, K, \lambda)/w(K)$ is monotone decreasing as a function of K.

Lemma III.3.—[Equilibrium unconstrained consumption] *Consider the function* $\xi^i(K) := \mathscr{C}^i(\delta_i, \gamma_i, K, \lambda)/w(K)$*:*

(i) *it is monotone decreasing as a function of* K *over* $]0, \check{K}_i]$, *with* $\lim_{K\to\check{K}_i} \xi^i(K) = 1-\lambda$*;*

(ii) *the function* $\xi^i(K)$ *tends to* $+\infty$ *for* $K \to 0$.

With imperfect financial markets, the establishment of the proof of the existence of a stationary equilibrium will require an extra assumption on the production technology.

Assumption T2. $[D_K F(K,1)\cdot K - \eta K]/[D_L F(K,1)]$ is bounded in a neighbourhood of 0.[2]

Proposition III.3.— [Existence of a stationary competitive equilibrium] *Under the previous set of assumptions,*

(i) *there does exist a stationary competitive equilibrium for the economy characterised by a value* $K^* \in]0, \check{K}_1[$*;*

(ii) *letting* $j \in \{1, \dots, n\}$ *be such that* $\check{K}_{j+1} \leqslant K^* < \check{K}_j$, *agents* $1, \dots, j$ *are unconstrained and consume according to* $c^i = \mathscr{C}(\delta_i, \gamma_i, K, \lambda)$ *whereas the agents* $j+1, \dots, n$ *are constrained and their consumptions are available as* $c^i = (1-\lambda)w$.

With perfect capital markets, the consumption function was decreasing in K for all unconstrained agents and cancels out for the constrained ones, a property that ensured the uniqueness of the equilibrium. In opposition to this and with imperfect financial markets, the uniqueness of the equilibrium cannot be established in the general case. Constrained agents now have a consumption level $(1-\lambda)w$, that increases with K and results in the non-monotonicity of aggregate consumption.

[2]This property holds for a Cobb-Douglas production function or for a C.E.S. production function when the elasticity of substitution is greater than one. Indeed and for $F(K,1) = K^\alpha$, $\alpha \in]0,1[$, $\lim_{K\to 0}[D_K F(K,1)K - \eta K]/D_L F(K,1) = \alpha/(1-\alpha)$ while $\lim_{K\to 0}[D_K F(K,1)K - \eta K]/D_L F(K,1) = 0$ for such a C.E.S production function.

Example III.3. The aim of this example is to establish how, within an economy populated by two agents $i = 1, 2$ and such that $\delta_1 > \delta_2$ holds together with $\gamma_1 > \gamma_2$, the equilibrium can be characterized by a configuration for which the relatively more patient agent 1 will be constrained whilst the relatively more impatient agent 2 will not. Only the most impatient of the agents will then be unconstrained. Letting again $u(c) = \ln(c)$ and $v(c) = c$, such an equilibrium is to be characterized by a value K such that:

$$\frac{1-\lambda}{2} D_L F(K,1) + \frac{[D_K F(K,1) + (1-\eta)]\delta_2 - 1}{\gamma_2} = F(K,1) - \eta K \tag{7}$$

with a value of K such that:

$$\frac{[D_K F(K,1) + (1-\eta)]\delta_2 - 1}{\gamma_2} > \frac{1-\lambda}{2} D_L F(K,1), \tag{8a}$$

$$\frac{[D_K F(K,1) + (1-\eta)]\delta_1 - 1}{\gamma_1} < \frac{1-\lambda}{2} D_L F(K,1), \tag{8b}$$

where these two above inequations respectively pictured an unconstrained agent 2 and a constrained agent 1.

In order to build such an equilibrium, one needs to let the analysis rest upon a value of $K > 0$ such that $K < \tilde{K}_2$, for $\tilde{K}_2$ that is uniquely defined by the holding of $D_K F(\tilde{K}_2, 1) + 1 - \eta = 1/\delta_2$. Fixing then δ_1 and δ_2, γ_1 and γ_2 are selected such that the equilibrium characterized by (7), (8a) and (8b) does correspond to K.

For that value of K, one selects γ_2 such that:

$$\frac{[D_K F(K,1) + (1-\eta)]\delta_2 - 1}{\gamma_2} = F(K,1) - \eta K - \frac{1-\lambda}{2} D_L F(K,1).$$

Such a value of γ_2 exists since

$$\begin{aligned} F(K,1) - \eta K - \frac{1-\lambda}{2} D_L F(K,1) &= \frac{1+\lambda}{2} D_L F(K,1) + K[D_K F(K,1) - \eta] \\ &> 0. \end{aligned}$$

Indeed, by assumption, $K < \tilde{K}_2 < \hat{K}$, one obtains $D_K F(K,1) - \eta > 0$. Further and for the appropriate value of γ_2, (8a) needs to be satisfied, or

$$\frac{1-\lambda}{2} D_L F(K,1) + \frac{1-\lambda}{2} D_L F(K,1) < F(K,1) - \eta K.$$

The holding of this inequality boils down to the satisfaction of $\lambda D_L F(K,1) + K[D_K F(K,1) - \eta] > 0$, that does prevail.

Finally, inequation (8b) will always be satisfied through the selection of large enough values for γ_1.

IV. Comparative Statics & Welfare Topics

IV.1 Comparative Statics

This section is aimed at understanding the dependency of the long-run equilibrium with respect to the parameters δ_i, γ_i and λ.

IV.1.1 The Configuration without Financial Constraints

For this configuration, the stationary state is unique and there exists $\zeta \in [1,n] \cap \mathbb{N}$ such that, for any $i > \zeta$, $c^i = 0$ and for any $i \leqslant \zeta$, $c^i > 0$. Further and for any $i \leqslant \zeta$, $c^i(K)$ decreases as a function of K. The following equation

$$\sum_{i=1}^{n} \mathscr{C}^i(\delta_i, \gamma_i, K, 1) = F(K, 1) - \eta K$$

then assumes a unique solution $K^* \in]0, \hat{K}[$, for $\hat{K}$ defined from $D_K F(\hat{K}, 1) = \eta$. It is also obtained that

$$K^* \in \,]\widetilde{K}_\zeta, \widetilde{K}_{\zeta+1}[.$$

But the occurrence of the condition $c^i = 0$ for $i > \zeta$ does solely proceed through the parameter δ_i and is independent of γ_i as it corresponds to $\delta_i R \leq 1$. From the properties of the function $\mathscr{C}^i$ listed through Lemma III.1, the following list of comparative statics results then becomes available:

Proposition IV.1.— *Let $\lambda = 1$:*

(i) *consider an agent $i = 1, 2, \ldots, n$ such that $i > \zeta$ and $c^{i*} = 0$, then any variation of his degree of self-control γ_i does not have any effect on the equilibrium steady state;*

(ii) *consider an agent $i = 1, 2, \ldots, n$ such that $i \leqslant \zeta$ and $c^{i*} > 0$, then either an infinitesimal* decrease *of γ_i or an infinitesimal* increase *in δ_i both translate into a* long-run *increase of the value of the capital stock.*

IV.1.2 The Configuration with Financial Constraints

Consider the competitive equilibrium $K^* \in]0, \check{K}_1[$ and assume that $K^* \in]\check{K}_\zeta, \check{K}_{\zeta+1}[$, $\zeta \in [1, n]$. Then and for any $i \geqslant \zeta + 1$,

$$\mathscr{C}^i(\delta_i, \gamma_i, K^*, \lambda) = (1 - \lambda) w(K^*)$$

and the associated individuals are constrained. In opposition to this and for any $i \leqslant \zeta$,

$$\mathscr{C}^i(\delta_i, \gamma_i, K^*, \lambda) > (1 - \lambda) w(K^*)$$

and the associated individuals are unconstrained.

Proposition IV.2.— *Assume that, for any λ, the equilibrium is unique and let $\lambda < 1$:*

(i) *consider an agent $i \in [1, n]$ such that $i > \zeta$: then an infinitesimal variation of his degree of impatience δ_i or of his degree of self-control γ_i does not have any effect on the equilibrium steady state;*

(ii) *consider an agent $i \in [1, n]$ such that $i \leqslant \zeta$ and $c^{i*} > 0$, then either an infinitesimal* decrease *of γ_i or an infinitesimal* increase *in δ_i both translate into a* long-run *increase of the value of the capital stock;*

(iii) *an increase in λ reduces the equilibrium steady state value of the capital stock K^*.*

Proof: By definition, K^* is a solution of the following equation:

$$F(K,1) - \eta K = \sum_{i=1}^{\zeta} c^i(K) + (n-\zeta)(1-\lambda)w(K),$$

or $H^\ell(K) = H^r(K,\gamma_i,\delta_i,\lambda)$, for $i = 1,\dots,k$. The uniqueness of the equilibrium steady state ensures that

$$D_K H^\ell(K^*) - D_K H^r(K^*,\gamma_i,\delta_i,\lambda) > 0.$$

Taking then advantage of the independence of $H^r(K,\gamma_i,\delta_i,\lambda)$ with respect to γ_i and δ_i for any $i \leqslant \zeta$, this establishes $\left(\mathrm{i}\right)$.

The obtention of $\left(\mathrm{ii}\right)$ results from the occurrence of $D_{\delta_i}C(\delta_i,\gamma_i,R,w,\lambda) > 0$, $D_{\gamma_i}C(\delta_i,\gamma_i,R,w,\lambda) < 0$ through Lemma III.1.

The obtention of $\left(\mathrm{iii}\right)$ in his turn results from the occurrence of $D_\lambda C(\delta_i,\gamma_i,R,w,\lambda) < 0$ through Lemma III.1. Q.E.D

The results of Proposition IV.2 have been reached through a uniqueness assumption for the stationary state. It is worth emphasizing that, even without such an assumption, the above results will have prevailed in a neighbourhood of $\lambda = 1$. Indeed, and for $\lambda = 1$, $D_K H^\ell(K^*) - D_K H^r(K^*,\gamma_i,\delta_i,1) > 0$, that establishes the robustness of the preceding conclusions.

According to $\left(\mathrm{iii}\right)$, an increase in λ will result in a decrease of the long-run value of the capital stock K^*. From $K^* < \widehat{K}$, this will in turn translate in a decrease of total consumption. But this will not necessarily correspond to a decrease in the individual consumption of any of the agents. For constrained individuals, consumption being given by $(1-\lambda)w(K)$, it will decrease with a rise in λ. But, and for unconstrained individuals, this may result into an increase of their consumption. The previously considered example that built from $u(c) = \ln(c)$ and $v(c)$ provides an enlightening illustration in this regard. The consumption of an unconstrained individual being available as $c^i = (\delta_i R - 1)/\gamma_i$, an increase in λ that entails a decrease in K^* will eventually translate into an increase in c^i.

IV.2 Welfare & Optimality

This section will first introduce the social optimum and then illustrate how it can be recovered trough a competitive equilibrium.

Assume that the economy is populated by n agents of parameters (δ_i,γ_i) where $1 > \delta_1 > \delta_2 > \cdots > \delta_n$. Letting $(\zeta_1,\dots,\zeta_n)$ denote the welfare weights put on any of the n agents, the program of the benevolent planer will then formulate along:

$$\max_{\{c_t^i,\omega_t^i\}} \sum_{i=1}^{n} \zeta_i \sum_{t=0}^{+\infty} (\delta_i)^t \left\{ u(c_t^i) + \gamma_i\left[v(c_t^i) - v(\omega_t^i)\right]\right\}$$

$$\text{s.t} \quad K_{t+1} = F(K_t,1) + (1-\eta)K_t - \sum_{i=1}^{n} c_t^i,$$

$$c_t^i \leqslant \omega_t^i, \quad K_0 \text{ given},$$

where ω_t^i singles out the maximal admissible value for the consumption of agent i and by date $t \geqslant 0$. The component of the planer's objective $v(c_t^i) - v(\omega_t^i)$ being however limited to assume non-positive values as a result of the holding of the constraint $c_t^i \leqslant \omega_t^i$, the optimal solution will be associated with the uniform satisfaction of $\omega_t^i = c_t^i$. The benevolent planer's program hence simplifies to the standard formulation of the optimal growth problem:

$$
\begin{aligned}
(\mathrm{P}) \quad & \max_{\{c_t^i, \omega_t^i\}} \sum_{i=1}^{n} \zeta_i \sum_{t=0}^{+\infty} (\delta_i)^t u(c_t^i) \\
& \text{s.t} \quad K_{t+1} = F(K_t, 1) + (1-\eta)K_t - \sum_{i=1}^{n} c_t^i, \\
& \qquad\quad K_0 \text{ given.}
\end{aligned}
$$

The optimal solution being denoted $\{c_t^{i*}, K_t^*\}$, $t \geqslant 0$, from Le Van & Vailakis [6], it will converge towards the stationary state of the modified golden rule $(c^{1*}, \ldots, c^{n*}, K^*)$ such that:

$$
\begin{aligned}
& D_K F(K^*, 1) + 1 - \eta = 1/\delta_1, \\
& c^{1*} = F(K^*, 1) - \eta K^*, \\
& c^{i*} = 0 \text{ for any } i > 1.
\end{aligned}
$$

It is then a standard argument to establish that a characterisation of the optimal trajectory is available from the following statement:

Lemma IV.1.— *The sequence $\{c_t^{i*}, K_t^*\}_{t \geqslant 0}$ depicts an optimal solution to the problem* (P) *if and only if there exists a sequence $\{\chi_t\}_{t \geqslant 0}$ that satisfies, for any $i = 1, 2, \ldots, n$ and any $t \geqslant 0$:*

$$
\begin{aligned}
& \zeta_i (\delta_i)^t D u(c_t^{i*}) = \chi_{t+1}, \\
& \chi_t = \chi_{t+1} [D_K F(K_t^*, 1) + 1 - \eta], \\
& K_{t+1}^* = F(K_t^*, 1) + (1-\eta)K_t^* - \sum_{i=1}^{n} c_t^{i*}, \\
& \lim_{t \to +\infty} \chi_t K_t^* = 0.
\end{aligned}
$$

The completion of a decentralization procedure for the first-best optimum will first rely upon the selection by any date $t \geqslant 0$ of the maximal value that is reachable for a level of consumption, i.e., ω_t^i . The component $v(c_t^i) - v(\omega_t^i)$ being negative for $c_t^i < \omega_t^i$, a decentralization scheme will proceed from imposing $\omega_t^i = c_t^{i*}$ for any $t \geqslant 0$. It is then noticed that this further corresponds to a sufficient condition to ensure the obtention of the first-best optimum that will be associated with an appropriate redistribution of the initial wealth of any of the individuals.

The establishment of the decentralization argument through Proposition IV.3 will first proceed from a preparatory lemma.

Lemma IV.2.— *The competitive economy with consumption constraints along $c_t^i \leqslant c_t^{i*}$ is fully characterized by sequences $\left(c_t^i, a_t^i, K_t, w_t, R_t\right)_{t \geqslant 0}$, $(\mu_t^i)_{t \geqslant 0}$ and $(\nu_t^i)_{t \geqslant 0}$ such that, for any $t \geqslant 0$:*

$$
\begin{aligned}
& (\delta_i)^t [D u(c_t^i) + \gamma_i D v(c_t^i)] = \mu_{t+1}^i + \nu_t^i, \\
& a_{t+1}^i = R_t a_t^i + w_t - c_t^i,
\end{aligned}
$$

$$
\begin{aligned}
&\mu_t^i = \mu_{t+1}^i R_t,\\
&\lim_{t\to+\infty} \mu_t^i a_t^i = 0, a_0^i \textit{ given},\\
&\nu_t^i = 0 \text{ for } c_t^i < c_t^{i*},\\
&\nu_t^i > 0 \text{ for } c_t^i = c_t^{i*},\\
&R_t = D_K F(K_t, 1) + 1 - \eta,\\
&w_t = D_L F(K_t, 1)/n,\\
&K_t = \sum_{i=1}^{n} a_t^i.
\end{aligned}
$$

Proposition IV.3.— *The social optimum can be decentralized by choosing at date $t = 0$ the initial wealth a_0^i of any of the agents—through a redistribution of the initial wealth—and by imposing $c_t^i \leqslant c_t^{i*}$ at any agent $i \in [1, n]$ and for any period $t \geqslant 0$.*

Surprisingly enough, Proposition IV.3 results in a set of conclusions that advocates communism. As soon as the agents are submitted to a temptation motive, the best admissible policy will be to severe their freedom of choice in consumption by imposing a constraint $c_t^i \leqslant c_t^{i*}$: any of the agents is to consume according to his actual needs.

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V. Proofs

V.1 Proof of Proposition III.1

The satisfaction of $R = D_K F(K, 1) + 1 - \eta$ and $w = D_L F(K, 1)/n$ directly results form equations (3h) and (3i) when they are considered at the equilibrium long-run steady state. In parallel to this, the consideration of (3g) allows for deriving $h = w/(R-1)$ whereas (3d) will hold for a stationary state. Equations (3a), (3b), (3c), (3e) and (3f) will then allow for obtaining Proposition III.1(i) and Proposition III.1(ii) through the following statement:

Lemma.— *Consider the optimal solution:*

(i) *a stationary optimal solution for an unconstrained individual corresponds to stationary values c^i and x^i such that:*

$$R\delta_i = \frac{Du(c^i) + \gamma_i Dv(c^i)}{Du(c^i) + \gamma_i (Dv(c^i) - Dv(x^i + \lambda h))},$$

where

a/ $x^i = (Rc^i - w)/(R - 1)$;

b/ *the satisfaction of $c^i < x^i + \lambda h$ boils down to $c^i > w(1 - \lambda)$ or $\gamma_i Dv[(1 - \lambda)w] < (\delta_i R - 1) Du[(1 - \lambda)w]$;*

c/ *the existence of such a solution requires* $R\delta_i > 1$ and implies $R > 1$;

(ii) *a stationary optimal solution for a constrained individual corresponds to stationary values c^i and x^i such that*

a/ $c^i = (1 - \lambda)w$;

b/ $x^i = [R(1 - \lambda)w - \lambda w/(R - 1)$;

c/ *the existence of such a solution requires the holding of $\nu^i > 0$, that writes down along: $\gamma_i Dv((1 - \lambda)w) > (\delta_i R - 1)Du((1 - \lambda)w)$.*

Proof: (i) For an unconstrained individual, $\nu_t^i = 0$ and from (3b) and (3c):

$$(*) \qquad R_{t+1}\delta_i [Du(c_{t+1}^i) + \gamma_i Dv(c_{t+1}^i) - \gamma_i Dv(x_{t+1}^i + \lambda h_{t+2})] = Du(c_t^i) + \gamma_i Dv(c_t^i).$$

Rearranging and considering stationary trajectories in (*), the conditions of the statement become available.

(ii) The basic form of the condition $\nu_i > 0$ lists as:

$$\nu^i = Du(c^i) + \gamma_i Dv(c^i) - \delta_i \mu^i R > 0,$$

for $\mu^i = Du(c^i) + \gamma_i Dv(c^i) - \gamma_i Dv(x^i + \lambda h)$, $c^i = x^i + \lambda h$, $\mu^i = Du(c^i)$, that gives, merging and rearranging, the condition of the statement. Q.E.D

Equation (4a) then corresponds to the resource constraints of the economy. It is derived by combining equations (3j), (3h), (3i) and (3a) when considered at the stationary state.

Equation (3j) allows for stating $\sum_{i=1}^n x^i = Rx+nw$ whilst, from equation (3a), $\sum_{i=1}^n x^i(R-1) = R\sum_{i=1}^n c^i - nw$. Finally eliminating $\sum_{i=1}^n x^i$ between these equations, it is derived that:

$$(R-1)K + nw = \sum_{i=1}^{n} c^i.$$

Making use of (3h) and (3i), equation (4a) is recovered. Q.E.D.

V.2 Proof of Lemma III.1

Proof: (i) From (5), the defining equation for a steady state consumption is available as:

$$Du(c^i) + \gamma_i Dv(c^i) = \frac{\delta_i R}{\delta_i R - 1}\gamma_i Dv\left[\frac{Rc^i - (1-\lambda)w}{R-1}\right],$$

a solution to the above equation such that $c > (1-\lambda)w$ is then sought for. It is worth noticing that this conveniently restates as $H^\ell(c^i,\gamma_i) = H^r(c^i,\delta_i,\gamma_i,R,w,\lambda)$, where the simultaneous occurrence of $D_{c^i}H^\ell(c^i,\gamma_i) < 0$ and $D_{c^i}H^r(c^i,\delta_i,\gamma_i,R,w,\lambda) > 0$ first ensures that there exists at most one solution. Dealing then with the existence issue and for $c^i \to (1-\lambda)w$, it is obtained that:

$$\begin{aligned} H^\ell(c^i,\gamma_i) &\to Du[(1-\lambda)w] + \gamma_i Dv[(1-\lambda)w], \\ H^r(c^i,\delta_i,\gamma_i,R,w) &\to \frac{\delta_i R}{\delta_i R - 1}\gamma_i Dv[(1-\lambda)w]. \end{aligned}$$

From the previous assumption, $\gamma_i Dv[(1-\lambda)w] < (\delta_i R - 1)Du[(1-\lambda)w]$, whence

$$H^\ell((1-\lambda)w,\gamma_i) > H^r((1-\lambda)w,\delta_i,\gamma_i,R,w,\lambda).$$

Conversely and for $c^i \to +\infty$, whereas $Du(c^i) \to 0$, $[Rc^i - (1-\lambda)w]/(R-1) > c^i$ as a result of the occurrence of $c^i > (1-\lambda)w$. This eventually results in

$$H^\ell(+\infty,\gamma_i) < H^r(+\infty,\delta_i,\gamma_i,R,w,\lambda),$$

whence the existence and uniqueness results for c^i.

(ii) Looking then for some comparative statics properties, it first emerges that $D_R C(\delta_i,\gamma_i,R,w,\lambda) > 0$ as a byproduct of $D_R H^r(c^i,\delta_i,\gamma_i,R,w) < 0$. Indeed

$$\begin{aligned} \mathrm{Sgn}\left[D_R\left(\frac{Rc^i - (1-\lambda)w}{R-1}\right)\right] &= \mathrm{Sgn}[c^i(R-1) - Rc^i + (1-\lambda)w] \\ &= \mathrm{Sgn}[(1-\lambda)w - c^i] \\ &< 0. \end{aligned}$$

The establishment that $D_w C(\delta_i,\gamma_i,R,w,\lambda) \geqslant 0$ then results from $D_w H^r(c^i,\delta_i,\gamma_i,R,w,\lambda) \leqslant 0$, having noticed that, for $\lambda = 1$, this simplifies to $D_w C(\delta_i,\gamma_i,R,w,\lambda) = 0$. In the same way, it is readily shown that $D_{\delta_i} C(\delta_i,\gamma_i,R,w,\lambda) > 0$ and $D_\lambda C(\delta_i,\gamma_i,R,w,\lambda) < 0$ are direct byproducts of $D_{\delta_i} H^r(c^i,\delta_i,\gamma_i,R,w,\lambda) < 0$ and $D_\lambda H^r(c^i,\delta_i,\gamma_i,R,w,\lambda) \geqslant 0$. Finally, the establishment of the occurrence of $D_{\gamma_i} C(\delta_i,\gamma_i,R,w,\lambda) < 0$ proceeds from the restatement of the defining equation for stationary values for c^i according to:

$$(**) \qquad \frac{R\delta_i}{R\delta_i - 1} = \frac{Du(c^i) + \gamma_i Dv(c^i)}{\gamma_i Dv[(Rc^i - (1-\lambda)w)/(R-1)]} \equiv G(c^i,\gamma_i,w,R).$$

The function $G(c^i, \gamma_i, w, R)$ being decreasing as a function of both γ_i and c^i, the occurrence of $D_{\gamma_i} C(\delta_i, \gamma_i, R, w, \lambda) < 0$ is established.

The expression $C(\delta_i, \gamma_i, R, w, \lambda)$ being monotone increasing as a function of R, it assumes a limit that is either finite or infinite. Firstly assuming that $\ell < +\infty$ and from equation $(**)$, it is obtained that:

$$1 = \frac{Du(\ell) + \gamma_i Dv(\ell)}{\gamma_i Dv(\ell)},$$

that is impossible for $\ell < +\infty$ and establishes $\ell = +\infty$. Q.E.D

V.3 Proof of Proposition III.2

Proof: (i)-(ii) In order to save on notations, one considers the function $c^i(K)$ as defined from:

$$\begin{cases} c^i(K) = \mathscr{C}(\delta_i, \gamma_i, K, 1) & \text{for} \quad K \in \left]0, \tilde{K}_i\right[; \\ c^i(K) = 0 & \text{for} \quad K \geqslant \tilde{K}_i. \end{cases}$$

The existence of an equilibrium is associated to the one of a value of K such that:

$$F(K, 1) - \eta K = \sum_{i=1}^{n} c^i(K).$$

Denoting this equation along $H^\ell(K) = H^r(K)$, as $1/\delta_1 > 1$, it is first obtained that $\tilde{K}_1 < \hat{K}$. The function $H^\ell(K)$ then emerges as being increasing over $]0, \tilde{K}_1[$. In opposition to this, the function $H^r(K)$ is decreasing over $]0, \tilde{K}_1[$ with $H^r(\tilde{K}_1) = 0$ and $\lim_{K \to 0} H^r(K) = +\infty$. These properties ensure the existence and the uniqueness of $K^* \in]0, \tilde{K}_1[$ such that $H^\ell(K^*) = H^r(K^*)$. Q.E.D

V.4 Proof of Lemma III.2

Proof: (i)-(ii) Uniqueness is established from noticing that the L.H.S. of (6) is non-decreasing as a function of K whilst the R.H.S. decreases as a function of K.

The existence argument in turn results from noticing that, for $K \to \tilde{K}_i$,

$$\gamma_i Dv\left[(1-\lambda) D_L F(K, 1)/n\right] \to \gamma_i Dv\left[D_L F(\tilde{K}_i, 1)/n\right] > 0,$$
$$\left\{\delta_i \left[D_K F(K, 1) + (1-\eta)\right] - 1\right\} Du\left[(1-\lambda) D_L F(K, 1)/n\right] \to 0.$$

In parallel to this and for $K \to 0$,

$$\gamma_i Dv\left[(1-\lambda) D_L F(K, 1)/n\right] \to \gamma_i Dv\left[(1-\lambda) D_L F(0, 1)/n\right],$$
$$\left\{\delta_i \left[D_K F(K, 1) + (1-\eta)\right] - 1\right\} Du\left[(1-\lambda) D_L F(K, 1)/n\right] \to +\infty.$$

Whence the existence result. Q.E.D

V.5 Proof of Lemma III.3

Proof: (i) From its very definition, $\xi^i(K)$ is a solution to:

$$\frac{\delta_i R(K) \gamma_i}{\delta_i R(K) - 1} = \frac{Du(\xi^i w(K)) + \gamma_i Dv(\xi^i w(K))}{Dv((R(K)\xi^i - (1-\lambda)) w(K)/(R(K) - 1))}$$

The L.H.S. of the preceding equation increases as a function of K whilst the R.H.S. may conveniently be reformulated along $H^u(K,\xi^i)/H^\ell(K,\xi^i)$. It is then remarked that H^u decreases with K whereas H^ℓ increases with both K and ξ^i, $\left[R\xi_i - (1-\lambda)\right]/\left(R-1\right)$ being indeed a decreasing function of R, whence the monotonicity property for ξ^i.

(ii) The function $\xi^i(K)$ being decreasing monotone as a function of K, its assumes a limit at zero that is either $\ell > 0$ or $+\infty$. Assuming first that it is finite and $\lim_{K\to 0}\xi^i(K) = \ell$, the above equation delivers:

$$\gamma_i = \frac{Du(w(0)\ell) + \gamma_i Dv(w(0)\ell)}{Dv(w(0)\ell)},$$

that is impossible $\left(\text{a contradiction}\right)$, whence $\ell = +\infty$. Q.E.D

V.6 Proof of Proposition III.4

Proof: (i) For a constrained individual i, it is convenient to extend the function $\xi^i(K)$ over the interval $[\check{K}_i, +\infty[$ with $\xi^i(K) = 1-\lambda$. The existence of an equilibrium is hence ensured by the one of a value of K such that:

$$F(K,1) - \eta K = \sum_{i=1}^{n} \xi^i(K)\frac{D_L F(K,1)}{n},$$

hence, making use of the Euler relationship,

$$\frac{D_K F(K,1)K - \eta K}{D_L F(K,1)} = \frac{1}{n}\sum_{i=1}^{n}[\xi^i(K) - 1].$$

or $H^\ell(K) = H^r(K)$. First and for $K \to 0$, $H^\ell(K) < H^r(K)$. Indeed, whilst $H^\ell(K)$ is bounded under Assumption T.2, it is obtained that $H^r(K) \to \infty$ as a direct corollary of Lemma III.3(ii).

Now, for K such that:

$$K = \check{K}_1 > \check{K}_2 > \cdots > \check{K}_n,$$

$\xi^i(K) = 1-\lambda$ for any $i = 1,\ldots,n$ and $H^r(\check{K}_1) = -\lambda$. Further, $D_K F(\check{K}_1,1)\check{K}_1 - \eta\check{K}_1 > 0$ since $\check{K}_1 < \hat{K}$, for $\hat{K}$ that satisfies $D_K F(\hat{K},1) = \eta$. This implies $H^\ell(\check{K}_1) > H^r(\check{K}_1)$. The existence of a $K \in]0,\check{K}_1[$ such that $H^\ell(K_1) = H^r(K_1)$ is established. Q.E.D

V.7 Proof of Lemma IV.2

Proof: The consumer program writes down along:

$$\max_{\{c_t^i\}} \sum_{t=0}^{+\infty} (\delta_i)^t \left\{u(c_t^i) + \gamma_i[v(c_t^i) - v(c_t^{i*})]\right\}$$

$$\text{s.t} \quad a_{t+1}^i = R_t a_t^i + w_t - c_t^i,$$

$$\lim_{t\to+\infty} a_t^i \Big/ \prod_{\tau=0}^{t} R_\tau \geqslant 0,$$

$$c_t^i \leqslant c_t^{i*}, \quad a_0^i \text{ given.}$$

The optimality conditions are obtained by first listing the Lagrangian:

$$L_t = (\delta_i)^t \left\{u(c_t^i) + \gamma_i[v(c_t^i) - v(c_t^{i*})]\right\} + \mu_{t+1}^i[R_t a_t^i + w_t - c_t^i] - \mu_t^i a_t^i + \nu_t^i(c_t^{i*} - c_t^i),$$

Whence:

$$(\delta_i)^t [Du(c_t^i) + \gamma_i Dv(c_t^i)] = \mu_{t+1}^i + \nu_t^i,$$
$$\mu_{t+1}^i = R_t \mu_t^i,$$
$$\lim_{t\to+\infty} \mu_t^i a_t^i = 0.$$

In parallel to this, the optimal behaviour of the competitive firm delivers the equilibrium values of the real wage rate and of the rental rate for the capital stock along $w_t = D_L F(K_t, 1)/n$ and $R_t = D_K F(K_t, 1) + 1 - \eta$. The equilibrium on the capital market being finally associated with the holding of $\sum_{i=1}^n a_t^i = K_t$ for any $t \geqslant 0$, the characterisation follows. Q.E.D

V.8 Proof of Proposition IV.3

Proof: The argument derives by proving that the sequence $\{c_t^{i*}, K_t^*\}, i = 1, \ldots, n$, satisfies the whole set of conditions that characterise a competitive equilibrium with constraints upon consumption for appropriately chosen values of μ_t^i and ν_t^i and for a fitting distribution of the initial wealths a_0^i. Letting $\mu_{t+1}^i = \chi_{t+1}/\zeta_i$ and $\nu_t^i = (\delta_i)^t \gamma_i Dv(c_t^i)$. Defining then the prices of the factors according to $w_t^* = D_L F(K_t^*, 1)/n$ and $R_t^* = D_K F(K_t^*, 1) + 1 - \eta$, the initial wealths of the agents are selected so as to satisfy:

$$a_0^i = \sum_{t=0}^{+\infty} \frac{c_t^{i*}}{\prod_{\tau=0}^t R_\tau^*} - \sum_{t=0}^{+\infty} \frac{w_t^*}{\prod_{\tau=0}^t R_\tau^*},$$

that is possible since

$$\begin{aligned}
\sum_{i=1}^n a_0^i &= \sum_{t=0}^{+\infty} \frac{\sum_{i=1}^n c_t^{i*} - n w_t^*}{\Pi_{\tau=0}^t R_\tau^*} \\
&= \sum_{t=0}^{+\infty} \frac{F\left(K_t^*, 1\right) - K_{t+1}^* + (1-\eta)K_t^* - F_L\left(K_t^*, 1\right)}{\Pi_{\tau=0}^t R_\tau^*} \\
&= \sum_{t=0}^{+\infty} \frac{R_t^* K_t^* - K_{t+1}^*}{\Pi_{\tau=0}^t R_\tau^*} \\
&= K_0^*.
\end{aligned}$$

The sequence $\{a_t^i\}$ is then recursively defined through

$$a_{t+1}^i = R_t a_t^i + w_t - c_t^{i*}.$$

Finally, and by the definition of a_0^i, one obtains:

$$\lim_{t\to+\infty} a_t^i \Big/ \prod_{\tau=0}^t R_\tau = 0,$$

that establishes Proposition IV.3. Q.E.D.